JOURNAL OF APPROXIMATION THEORY 4, 137–146 (1971)

On Simultaneous Chebyshev Approximations

DAVID A. SPRECHER

Department of Mathematics, University of California,
Santa Barbara, California 93106

Communicated by T. J. Rivlin

Received August 1, 1969; Revised September 19, 1969

1. STATEMENT OF THE PROBLEM

Consider the space $\mathscr{C}$ of continuous real-valued functions with domain [0, 1]. For a given function $f \in \mathscr{C}$ let $E_j(f)$ represent its degree of Chebyshev approximation by algebraic polynomials of degree $\leqslant j$. With each polynomial p_j with $\deg p_j = j$, we associate the class

$$\mathscr{E}(p_j) = \{f \in \mathscr{C} \mid E_j(f) = \| f - p_j \|\}.$$

T. J. Rivlin [1] has raised the problem of characterizing the n-tuples of polynomials $\{p_0, p_1, ..., p_{n-1}\}$ with $\deg p_j = j$ for $j = 0, 1, ..., n-1$, such that

$$\mathscr{E}(p_0, p_1, ..., p_{n-1}) = \bigcap_{j=0}^{n-1} \mathscr{E}(p_j) \neq \varnothing.$$

He has shown that for this to be true, it is necessary for every difference $p_m - p_k$, $0 \leqslant k < m \leqslant n-1$ to change sign not less than $k+1$ times in the interval [0, 1]. Necessary and sufficient conditions for the case $n = 3$ are as follows:

THEOREM 1. *Let*

$$\begin{aligned} p_0(x) &= a_0, \\ p_1(x) &= b_0 + b_1 x \qquad (b_1 \neq 0), \\ p_2(x) &= c_0 + c_1 x + c_2 x^2 \qquad (c_2 \neq 0), \end{aligned}$$

be given polynomials. Then $\mathscr{E}(p_0, p_1, p_2) \neq \varnothing$ if and only if the following conditions are met:

(A) *There are points $0 < x_1 < x_2 < 1$ such that*

$$p_1(x_j) = p_2(x_j) \qquad \textit{for} \quad j = 1, 2.$$

(B) *If* $x_2 \leqslant x_1 + |b_1/c_2|$, *then*

$$p_1\left(\frac{x_1}{2}\right) < a_0 < \frac{1}{2}\,[p_2(x_1) + \max_{x_2 \leqslant x \leqslant 1} p_2(x)], \qquad (b_1 > 0,\ c_2 < 0),$$

$$p_1\left(\frac{x_2+1}{2}\right) < a_0 < \frac{1}{2}\,[p_2(x_2) + \max_{0 \leqslant x \leqslant x_1} p_2(x)], \qquad (b_1 < 0,\ c_2 < 0),$$

$$\frac{1}{2}\,[p_2(x_1) + \min_{x_2 \leqslant x \leqslant 1} p_2(x)] < a_0 < p_1\left(\frac{x_1}{2}\right), \qquad (b_1 < 0,\ c_2 > 0),$$

$$\frac{1}{2}\,[p_2(x_2) + \min_{0 \leqslant x \leqslant x_1} p_2(x)] < a_0 < p_1\left(\frac{x_2+1}{2}\right), \qquad (b_1 > 0,\ c_2 > 0).$$

If $x_2 > x_1 + |b_1/c_2|$, *let* $\xi_2 \leqslant \xi_1$ *be the solutions of the equation*

$$p_2(x) = p_1(x) + \min\left[p_1(0) - p_2(0), p_2\left(-\frac{c_1}{2c_2}\right) - p_1\left(-\frac{c_1}{2c_2}\right), p_1(1) - p_2(1)\right] \quad \textit{when} \quad c_2 < 0,$$

$$p_2(x) = p_1(x) + \max\left[p_1(0) - p_2(0), p_2\left(-\frac{c_1}{2c_2}\right) - p_1\left(-\frac{c_1}{2c_2}\right), p_1(1) - p_2(1)\right] \quad \textit{when} \quad c_2 > 0.$$

Then,

$$p_1\left(\frac{x_1}{2}\right) < a_0 < \frac{1}{2}\,[p_2(x_1) + p_2(\xi_1)], \qquad (b_1 > 0,\ c_2 < 0),$$

$$p_1\left(\frac{x_2+1}{2}\right) < a_0 < \frac{1}{2}\,[p_2(x_2) + p_2(\xi_2)], \qquad (b_1 < 0,\ c_2 < 0),$$

$$\frac{1}{2}\,[p_2(x_1) + p_2(\xi_1)] < a_0 < p_1\left(\frac{x_1}{2}\right), \qquad (b_1 < 0,\ c_2 > 0),$$

$$\frac{1}{2}\,[p_2(x_1) + p_2(\xi_2)] < a_0 < p_1\left(\frac{x_2+1}{2}\right), \qquad (b_1 > 0,\ c_2 > 0).$$

We mention in passing that the condition $x_2 \leqslant x_1 + |b_1/c_2|$ is equivalent to $x_2 \leqslant -(c_1/2c_2)$ when $b_1c_2 < 0$, and to $x_1 \geqslant -(c_1/2c_2)$ when $b_1c_2 > 0$; the condition $x_2 > x_1 + |b_1/c_2|$ is similarly related to $-(c_1/2c_2)$. This is seen by observing that $\xi = (x_1 + x_2)/2 = (b_1 - c_1)/2c_2$, ξ being the point at which $|p_1(x) - p_2(x)|$ assumes its maximum for $x_1 \leqslant x \leqslant x_2$. With the notation

$$p_j{}^* = -\operatorname{sgn} c_2 \cdot p_j,$$

$$A_1 = [x_2, 1], \qquad A_2 = [0, x_1],$$

$$a_2 = \frac{x_2+1}{2}, \qquad a_1 = \frac{x_1}{2},$$

$$p_2{}^*(\xi_{j1}) = \max_{x \in A_j} p_2{}^*(x), \qquad \xi_{j2} = \xi_j,$$

condition (B) can be condensed to the form

$$p_1{}^*(a_j) < p_0{}^*(x) < \tfrac{1}{2}[p_1{}^*(x_j) + p_2{}^*(\xi_{jk})],$$

where

$$j = \begin{cases} 1 & \text{when} \quad b_1c_2 < 0, \\ 2 & \text{when} \quad b_1c_2 > 0, \end{cases} \qquad k = \begin{cases} 1 & \text{when} \quad x_2 \leqslant x_1 + \left| \dfrac{b_1}{c_2} \right|, \\ 2 & \text{when} \quad x_2 > x_1 + \left| \dfrac{b_1}{c_2} \right|. \end{cases}$$

It suffices to carry out the proof for the case $b_1 > 0$ and $c_2 < 0$, because the other cases reduce to this one by reflections. Specifically, we let

$$\begin{array}{llll} x = 1 - v \ \text{ and } \ y = w & \text{when} & b_1 < 0 \ \text{ and } \ c_2 < 0, \\ x = v \ \text{ and } \ y = -w & \text{when} & b_1 < 0 \ \text{ and } \ c_2 > 0, \\ x = 1 - v \ \text{ and } \ y = -w & \text{when} & b_1 > 0 \ \text{ and } \ c_2 > 0. \end{array}$$

We shall further restrict the proof to the case $x_2 \leqslant x_1 + | b_1/c_2 |$, because the proof in the case $x_2 > x_1 + | b_1/c_2 |$ runs parallel to the former and involves no new ideas. We shall actually prove

THEOREM 2. *Let $p_0(x) = a_0$, $p_1(x) = b_0 + b_1x$ $(b_1 > 0)$, and $p_2(x) = c_0 + c_1x + c_2x^2$ $(c_2 < 0)$ be given polynomials, the last two intersecting at a point $x_2 \leqslant -(c_1/2c_2)$, $0 \leqslant x_2 \leqslant 1$. Then the following statements are equivalent:*

(C) $\mathscr{E}(p_0, p_1, p_2) \neq \varnothing$.

(D) *The polynomials $p_1(x)$ and $p_2(x)$ intersect also at a point x_1, where $0 < x_1 < x_2 < 1$, $x_2 \leqslant x_1 + | b_1/c_2 |$, and*

$$p_1\left(\frac{x_1}{2}\right) < a_0 < \frac{1}{2}\,[p_2(x_1) + \max_{x_2 \leqslant x \leqslant 1} p_2(x)].$$

(E) *There are constants*

$$\alpha_0 > \alpha_1 > \alpha_2 \geqslant 0$$

and points

$$0 \leqslant t_1 < t_2 \leqslant t_3 < t_4 \leqslant 1$$

such that if

$$p_j{}^-(x) = p_j(x) - \alpha_j$$

and

$$p_j{}^+(x) = p_j(x) + \alpha_j,$$

then

(E-1) $p_1^-(t_j) = p_2^-(t_j)$, *for* $j = 1, 4$,

(E-2) $p_1^+(t_j) = p_2^+(t_j)$, *for* $j = 2, 3$,

(E-3) $p_2^-(0) \leqslant p_0^-(x) \leqslant p_1^-(t_1)$,

(E-4) $p_1^+(t_2) \leqslant p_0^+(x) \leqslant \max_{t_3 \leqslant x \leqslant 1} p_2^+(x)$,

(E-5) $\| q^- - p_j \| \leqslant \alpha_j$ *and* $\| q^+ - p_j \| \leqslant \alpha_j$, *for* $j = 1, 2$, *where*

$$q^-(x) = \max\{p_0^-(x), p_1^-(x), p_2^-(x)\},$$
$$q^+(x) = \min\{p_0^+(x), p_1^+(x), p_2^+(x)\}.$$

The reasoning of the proof yields a corresponding result when polynomials are replaced by an arbitrary Chebyshev system. It seems to be unsuitable for attacking Rivlin's problem for $n > 2$ because of the large number of cases arising. We prove, however,

THEOREM 3. *If* $f \in \mathscr{E}(p_0, p_1, \ldots, p_{n-1})$, *then*

$$E_0(f) > E_1(f) > \cdots > E_{n-1}(f).$$

2. HEURISTIC CONSIDERATIONS

Certain *a priori* facts hold when we suppose that $f \in \mathscr{E}(p_0, p_1, p_2)$. Namely, there exist constants $\alpha_0 > \alpha_1 > \alpha_2 \geqslant 0$ satisfying:

(F) The function $f(x)$ meets each of the polynomials $p_j^-(x)$ and $p_j^+(x)$.

(G) $q^-(x) \leqslant f(x) \leqslant q^+(x)$.

(H) The polynomials $p_j^-(x)$ intersect pairwise, as do the polynomials $p_j^+(x)$. Properties (F) and (G) are evident.

To verify (H) we note that if, say, $p_0^-(x) < p_1^-(x)$, then $p_0^-(x) < q^-(x) \leqslant f(x)$, thereby contradicting (F).

We observe that $\mathscr{E}(p_0, p_1, p_2)$ cannot contain linear functions. If u is a second-degree polynomial in $\mathscr{E}(p_0, p_1, p_2)$, then $u \in \mathscr{E}(p_2)$, and this implies that $u = p_2$ and hence $\alpha_2 = 0$. Conversely, if $\alpha_2 = 0$ then p_2 is the only member of $\mathscr{E}(p_0, p_1, p_2)$. Thus, we will henceforth assume that $\alpha_2 > 0$.

Consider Fig. 1 below. Notice, in particular, the triangle $T_1T_2T_4$ and the relative position of the segment R_1R_2. Loosely speaking, our proof consists of showing that a similar situation is both necessary and sufficient for $\mathscr{E}(p_0, p_1, p_2)$ to be nonempty.

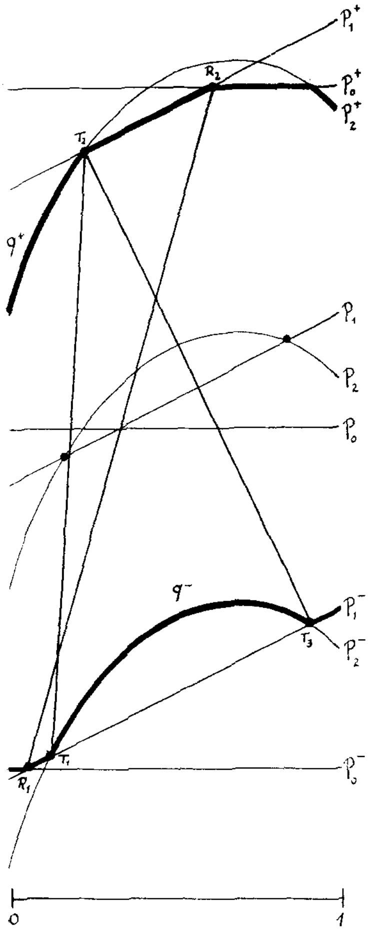

FIG. 1

3. PROOF THAT (C) IMPLIES (E)

Let us put

$$r^-(x) = \max\{p_1^-(x), p_2^-(x)\},$$
$$r^+(x) = \min\{p_1^+(x), p_2^+(x)\}.$$

We shall first show that the polynomials $p_1^-(x)$ and $p_2^-(x)$ must intersect at two distinct points of [0, 1]. If, on the contrary, they intersect at a single point t_1, then

$$r^-(x) = \begin{cases} p_1^-(x) & (0 \leqslant x \leqslant t_1), \\ p_2^-(x) & (t_1 \leqslant x \leqslant 1). \end{cases}$$

According to property (H), the polynomials $p_1^+(x)$ and $p_2^+(x)$ intersect in at least one point. Let $p_1^+(t_2) = p_2^+(t_2)$. If $t_2 \leqslant t_1$, then

$$r^+(x) = \begin{cases} p_2^+(x) & (0 \leqslant x \leqslant t_2), \\ p_1^+(x) & (t_2 \leqslant x \leqslant 1), \end{cases}$$

and we see that $f(x) - p_2(x) = E_2(f)$ only when $t_1 \leqslant x \leqslant 1$, whereas $f(x) - p_2(x) = -E_2(f)$ only when $0 \leqslant x \leqslant t_2$. This means, of course, that there are only two points in [0, 1] at which the difference $f(x) - p_2(x)$ equals, with alternating signs, to $E_2(f)$. As a result, $f \notin \mathscr{E}(p_2)$. On the other hand, when $t_2 > t_1$, when $r^+(x) < p_1^+(x)$ for $0 \leqslant x \leqslant t_2$. This implies that $f(x) - p_1(x) = E_1(f)$ only when $0 \leqslant x \leqslant t_1$ whereas $f(x) - p_1(x) = -E_1(f)$ only when $t_2 \leqslant x \leqslant 1$, and it follows that $f \notin \mathscr{E}(p_1)$. Thus, $f \notin \mathscr{E}(p_1, p_2)$, and therefore there are points t_1 and t_4 as asserted in (E-1).

Now, from property (H) we know that there is some point t_3 such that $p_1^+(t_3) = p_2^+(t_3)$. Suppose $0 \leqslant t_3 \leqslant t_1$. Then a simple geometric argument shows that $q^+(x) = p_1^+(x) < p_2^+(x)$ for all points $t_1 < x < t_2$, whereas $q^-(x) = p_1^-(x) > p_2^-(x)$ whenever $0 \leqslant x \leqslant t_1$ or $t_2 \leqslant x \leqslant 1$. It follows that there are at most three consecutive points in the interval [0, 1] at which the difference $f(x) - p_2(x)$ equals, with alternating signs, to $E_2(f)$. Once more we conclude that $f \notin \mathscr{E}(p_2)$, and so (E-2) is seen to hold.

The left inequality in (E-3) holds; for if $p_0^-(x) < p_2^-(0)$, then $p_0^-(x)$ does not intersect $q^-(x)$, thereby contradicting (H). To show that the right inequality holds, suppose $p_1^-(t_1) < p_0^-(x)$. Then $q^-(x) > p_1^-(x)$ for $0 \leqslant x < t_4$. Since $q^+(x) < p_1^+(x)$ for $t_3 < x \leqslant 1$, it follows that there are only two points at which $f(x) - p_1(x) = E_1(f)$ with alternating signs, implying that $f \notin \mathscr{E}(p_1)$. This establishes (E-3).

To establish (E-4) we note that if $p_0^+(x) < p_1^+(t_2)$, then $p_1^+(x)$ does not intersect $q^+(x)$, and if $\max_{t_3 \leqslant x \leqslant 1} p_2^+(x) < p_0^+(x)$, then $p_0^+(x)$ does not intersect $q^+(x)$. In either case, (H) is not satisfied and hence (E-4) is true.

Finally, the necessity of condition (E-5) is clear, for if, say, $\| q^- - p_1 \| > \alpha_1$, then $q^-(x) \leqslant f(x)$ implies that $\| f - p_1 \| > \alpha_1 = \| f - p_1 \|$.

4. Proof That (E) Implies (D)

The existence of the stipulated points $0 < x_1 < x_2 < 1$ in which $p_1(x)$ and $p_2(x)$ intersect follows at once. In particular, we have the inequalities

$$t_1 < x_1 < t_2 \leqslant t_3 < x_2 < t_4 .$$

Using the definitions of $p_j^-(x)$ and $p_j^+(x)$, the inequalities in (E-3) and (E-4) can be written in the form

$$b_0 - \alpha_1 \leqslant a_0 - \alpha_0 \leqslant b_0 + b_1 t_1 - \alpha_1$$

and

$$b_0 + b_1 t_2 + \alpha_1 \leqslant a_0 + \alpha_1 \leqslant \max_{t_3 \leqslant x \leqslant 1} p_2(x) + \alpha_2 .$$

These inequalities combine into

$$b_0 + b_1 \frac{t_2}{2} \leqslant a_0 \leqslant \frac{1}{2} [(b_0 + b_1 t_1) + \max_{t_3 \leqslant x \leqslant 1} p_2(x) - (\alpha_1 - \alpha_2)]. \tag{1}$$

We have

$$b_0 + b_1 \frac{t_2}{2} = p_1\left(\frac{t_2}{2}\right) > p_1\left(\frac{x_1}{2}\right) \qquad \text{because} \quad x_1 < t_2 ,$$

$$b_0 + b_1 t_1 = p_1(t_1) < p_1(x_1) \qquad \text{because} \quad t_1 < x,$$

and

$$\max_{t_3 \leqslant x \leqslant 1} p_2(x) - (\alpha_1 - \alpha_2) = \max_{x_2 \leqslant x \leqslant 1} p_2(x) - (\alpha_1 - \alpha_2) < \max_{x_2 \leqslant x \leqslant 1} p_2(x)$$

since $\alpha_1 - \alpha_2 > 0$. The inequalities (1) thus imply

$$p_1\left(\frac{x_1}{2}\right) < a_0 < \frac{1}{2} [p_1(x_1) + \max_{x_2 \leqslant x \leqslant 1} p_2(x)],$$

as was to be demonstrated.

5. Proof That (D) Implies (E)

According to condition (D) there is a constant $\beta_0 > 0$ such that

$$b_0 < a_0 - \beta_0 < p_1(x_1) \tag{2}$$

and

$$p_1(x_1) < a_0 + \beta_0 < \max_{x_2 \leqslant x \leqslant 1} p_2(x). \tag{3}$$

Let u_1 and u_2 satisfy in $[x_1 , x_2]$ the equations

$$p_2(u_1) = a_0 - \beta_0$$

and

$$p_2(u_2) = a_0 + \beta_0 ,$$

and let

$$0 < \beta_1 < \min\{b_0 - c_0 , p_1(u_1) - p_2(u_1), p_2(u_2) - p_1(u_2), \max_{x_2 \leqslant x \leqslant 1} p_2(x) - p_2(u_2)\}, \tag{4}$$

so that

$$\beta_1 < p_2(\xi) - p_1(\xi),$$

where $\xi = (x_1 + x_2)/2$. Letting α_2 stand for a positive constant to be specified subsequently, we put

$$\alpha_0 = \beta_0 + \beta_1 + \alpha_2 ,$$

and

$$\alpha_1 = \beta_1 + \alpha_2 .$$

With these constants we define the polynomials $p_j^-(x) = p_j(x) - \alpha_j$ and $p_j^+(x) = p_j(x) + \alpha_j$, and (H) is readily seen to be satisfied. Since $\alpha_1 > \alpha_2$, there are points $0 \leqslant t_1 < x_1 < t_2 \leqslant t_3 < x_2 < t_4 \leqslant 1$ for which (E-1) and (E-2) hold.

Let us now show that (E-3) and (E-4) are valid.

By (4), $c_0 \leqslant b_0 - \beta_1$, and according to (2), $b_0 < a_0 - \beta_0$. Hence,

$$\begin{aligned} p_2^-(0) &= c_0 - \alpha_2 \leqslant b_0 - (\beta_1 + \alpha_2) < a_0 - (\beta_0 + \beta_1 + \alpha_2) \\ &= a_0 - \alpha_0 = p_0^-(x). \end{aligned}$$

Again by (4), $p_2(u_1) < p_2(t_1)$, so that

$$\begin{aligned} p_0^-(x) &= a_0 - \beta_0 - \beta_1 - \alpha_2 = p_2(u_1) - \alpha_2 - \beta_1 \\ &= p_2^-(u_1) - \beta_1 < p_2^-(u_1) < p_2^-(t_1) = p_1^-(t_1), \end{aligned}$$

and hence (E-3) is established.

To verify (E-4), we observe that $t_2 < u_2$ by our choice of β_1, and, since $b_1 > 0$, this implies that

$$p_1^+(t_2) < p_1^+(u_2) < p_2^+(u_2) = a_0 + \beta_0 + \alpha_2 < a_0 + \beta_0 + \beta_1 + \alpha_2 = p_0^+(x).$$

Finally,

$$p_0^+(x) = p_2^+(u_2) + \beta_1 < \max_{x_2 \leqslant x \leqslant 1} p_2(x) = \max_{t_3 \leqslant x \leqslant 1} p_2(x)$$

since the maximum of $p_2(x)$ is assumed to be attained at x_2 or to its right.

According to what has just been proved, we have the situation described in Fig. 1. We now determine α_2 to satisfy (E-5), which completes the proof.

6. Proof That (E) Implies (C)

Fix four points u_j such that $t_1 < u_1 < u_2 < u_3 < u_4 < t_2$, and let g be the piecewise linear function on $[t_1, t_2]$ with vertices

$$\begin{matrix} (t_1, q^-(t_1)), & (u_1, q^+(u_1)), & (u_2, q^-(u_2)), \\ (u_3, q^+(u_3)), & (u_4, q^-(u_4)), & (t_4, q^+(t_4)) \end{matrix}$$

(in this connection see [2]). Put

$$f(x) = \begin{cases} q^-(x), & 0 \leqslant x \leqslant t_1, \\ g(x), & t_1 \leqslant x \leqslant t_2, \\ q^+(x), & t_2 \leqslant x \leqslant t_3, \\ \dfrac{t_4 - x}{t_4 - t_3} q^+(t_3) + \dfrac{x - t_3}{t_4 - t_3} q^-(t_4), & t_3 \leqslant x \leqslant t_4, \\ q^-(x), & t_4 \leqslant x \leqslant 1. \end{cases}$$

Then f is continuous and we assert that $f \in \mathscr{E}(p_0, p_1, p_2)$.

We first observe that, indeed,

$$q^-(x) \leqslant f(x) \leqslant q^+(x). \tag{5}$$

Owing to (E-3), $q^-(0) = p_0^-(0)$, and owing to (E-4) there is a point u, $t_2 \leqslant u \leqslant t_3$, such that $q^+(u) = p_0^+(u)$. These facts, together with (E-5) and (5), show that $f \in \mathscr{E}(p_0)$.

Next, we recall that $q^-(t_1) = p_1^-(t_1)$, $q^+(t_2) = p_1^+(t_2)$, and $q^-(t_4) = p_1^-(t_4)$. Since $t_1 < t_2 < t_4$, it follows again from (E-5) and (5) that $f \in \mathscr{E}(p_1)$.

Finally, we note that $q^-(x) = p_2^-(x)$ and $q^+(x) = p_2^+(x)$ for $t_1 \leqslant x \leqslant t_2$. The difference $g(x) - p_2(x)$ changes sign five times, and therefore $f \in \mathscr{E}(p_2)$.

This completes the proof of Theorem 2. Theorem 1 follows from the remarks preceding Theorem 2.

7. Proof of Theorem 3

We first note that for $0 \leqslant j \leqslant n - 1$,

$$E_j(f) = \inf_{\deg p = j} \| f - p \| = \inf_{\deg p \leqslant j} \| f - p \|.$$

As a consequence, we have that if $f \in \mathscr{E}(p_k, p_m)$ and $0 \leqslant k < m$ then $E_k(f) \geqslant E_m(f)$. But if $E_k(f) = E_m(f)$, then by the uniqueness of best approximations $p_k = p_m$, so that $\deg p_m = k < m$. This, however, contradicts the definition of $\mathscr{E}(p_k, p_m)$; hence $E_k(f) > E_m(f)$, and the theorem follows.

Acknowledgment

The author is indebted to Z. Ziegler and D. Amir for pointing out an error in the original formulation of Theorem 1. The above version of the proof of Theorem 3 is due to T. J. Rivlin.

REFERENCES

1. T. J. RIVLIN, Proc. Coll. Abstr. Spaces Approx. (Oberwolfach, July 1968), Birkhänsen Verlag, Ed. P. L. Butzer – B. Sz. Nagy, 1970.
2. D. A. SPRECHER, Simultaneous best approximations with two polynomials, *J. Approximation Theory* **2** (1969), 384–388.